A Fairly Complete Qualitative Analysis of a Discrete SIR Epidemic Model – Local Stability and generic codimension 1 Bifurcation

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Consider the following discrete SIR model with the susceptible population S assumed to follow the logistic growth with the growth rate $r > 0$ and the infectious force $\frac{\beta SI}{1+aS}$ from [4]:

$$S_{n+1} = rS_n(1 - S_n) - \frac{\beta S_n I_n}{1 + aS_n}, \quad I_{n+1} = \frac{\beta S_n I_n}{1 + aS_n} + (1 - \mu - \gamma)I_n, \quad R_{n+1} = \gamma I_n + (1 - \lambda)R_n$$

where the variables S, I and R are the populations of susceptible, infectious and recovered per capita respectively, and the parameters $a > 0$ measures the inhibitory effect, $\gamma > 0$ is the natural recovery rate of the infective individuals, $\mu > 0$ and $\lambda > 0$ are death rates of infective and recovered. In [4] the authors presented some numerical simulations, indicating local stability of fixed points and bifurcation to periodic doubling and limit cycles, but it is short of rigorous analysis, which in turn results in in-correct statement of the limit cycle in one example. In this presentation we give a complete qualitative analysis for non-negative parameters $a, \beta, K := \mu + \gamma$. Throughout our investigation we assume $K < 2$ and focus on the non-negative solutions. Moreover for simplicity stability means asymptotical stability. Since the last equation for R is not involved in the first two, our analysis is centered for the S and I, except that the initial values satisfies $S_0 + I_0 + R_0 = 1$ with $I_0 > 0$. The results are summarized below.

Local stability of fixed points

There are three fixed points on the SI-phase plane: $O = (0, 0)$, $E_0 = (\frac{r-1}{r}, 0)$ and the endemic $E_1 = (\frac{K}{\beta - aK}, \frac{r-1}{\beta - aK} - \frac{rK}{(\beta - aK)^2})$. The first trivial point O means extinction of all populations which is biologically not interesting. The disease free E_0 has physical meaning if $r > 1$ and the endemic fixed point E_1 is of physical interest if $\beta - aK > 0$ and $\beta > \beta_0 = \frac{K(r+a(r-1))}{r-1}$. For understanding complicated dynamical behavior of this system we include the analysis of the trivial fixed point O. By standard technique of linearization ([3]) and the Routh test for localizing polynomial zeros [2] we show that

(i) The trivial fixed point O is locally stable if $r < 1$;

(ii) the disease free fixed pint E_0 is locally stable if $1 < r < 3$ and $\beta < \beta_0$; and

(iii) the endemic fixed point E_1 is locally stable if $1 < r \leq 3$ and $\beta_0 < \beta < \beta_2$, or $3 < r < r_{\max}$ and $\beta_1 < \beta < \beta_2$, where $\beta_2 = \frac{1}{2}\left(a(2K-1) + \frac{r(K+1)}{r-1} + \sqrt{a^2 + \frac{2ar(3K-1)}{r-1} + \frac{r^2(K+1)}{(r-1)^2}}\right)$, $\beta_1 = \frac{1}{2}\left(\frac{K(2a(3+K(r-1)-r)+(K+1)r)}{4+K(r-1)} + \sqrt{\frac{K^2((K+2)^2r^2+4a^2(r+1)^2+4ar(14-5K-2r+3Kr))}{(4+K(r-1))^2}}\right)$ and $r_{\max} = \frac{1}{2}\sqrt{\frac{16a^2+88aK-32a+25K^2+40K+16}{K^2}} + \frac{4a+5K+4}{2K}$, at which the curve $\beta = \beta_1$ and $\beta = \beta_2$ in the parameter space intersects.

(Generic) co-dimension 1 bifurcations

It turns out that the systems at hand has very rich dynamics . With help of Mathematica we are able to find analytic expressions of parameter relations where bifurcations from the fixed points take place. Most of our results are based on the theory in [3] and analysis of eigenvalues to the Jacobian matrix. For precise definitions of bifurcation types we refer to [3].

Bifurcations from O: Since $K < 2$, O is a saddle point of $r > 1$ and a $r = 1$ there is a fold bifurcation. i.e. the stable O loses stability to other fixed points.

Bifurcations from E_0: There is a flip bifurcation if $r = 3$ and $\beta < \beta_0$, that is, the fixed point loses stability and period doubling take place. We have proved that it is generic and period doubling is stable. At $\beta = \beta_0$ for all $1 < r < 3$, or $r = 1$ for $\beta < \beta_0$ there is a fold bifurcation and E_0 becomes unstable.

Bifurcations from E_1: If $1 < r < 3$ and $\beta = \beta_0$ there is a fold bifurcation. Since it can be shown that the Jacobian matrix has a pair complex conjugate eigenvalues on $\beta = \beta_2$ for $1 < r < r_{\max}$ and two distinct real eigenvalues for $r > r_{\max}$, and on $\beta = \beta_1$ for $3 < r \neq r_{\max}$. This implies that we do not expect any Neimark-Sacker bifurcations (the eigenvalues are on the unit circle, the Hopf-type of bifurcation in discrete time) in the latter two cases. In the former case we show that there are Neimark-Sacker bifurcations except $r = \bar{r} = \frac{1}{2}\sqrt{\frac{4a^2+20aK-8a+9K^2+12K+4}{K^2}} + \frac{2a+3K+2}{2K}$. Numerical simulations show that stable limit cycle exist. We also found that there is a flip bifurcation on $\beta = \beta_1$ for $3 < r < r_{\max}$. Much likely this is an unstable period doubling because no such a dynamics has been observed in our numerical experiments.

Some results on co-dimension 2 bifurcations

We have proved that co-dimension 2 bifurcations take place for some parameters but we have not yet shown that they are generic. For the disease free fixed point we proved that if $r = 1$ and $\beta = \beta_0$ or $r = 3$ and $\beta = \beta_0$ there is a fold-flip bifurcation.

For the endemic fixed point E_1 the situation is more complicated. We have found the following types of bifurcations according to positions of the eigenvalues to the Jacobian matrix.

- For $r = 3$ and $\beta = \beta_0$ there is a fold-flip.
- For $r = r_{\max}$ and $\beta = \beta_2$ there is a 1:2 resonance (double eigenvalues at -1).
- For $r = \bar{r}(> 3$ on the curve $\beta = \beta_2$ there is a 1:4 resonance (the eigenvalues are $e^{\pm i\frac{\pi}{2}}$).
- there is no 1:1 resonance, no 1:3 resonance (the eigenvalues are $e^{\pm i\theta}$ with $\theta = \frac{3\pi}{2}$).

Period doubling and other periods.

We have seen that when $r = 3$ and $\beta < \beta_0$ there is a stable period doubling. Furthermore use the factor that the I-equation is linear in I, we prove that around $I = 0$ the dynamics of S dominates and S. And dynamics of S towards the celebrated Feigenbaum attractor. This means not only do we find the period doubling when $3 < r < \sqrt{3} + 2$ for $\beta < \beta_0$ but also period 3 and all other periods according to Sarkovskii's Theorem [1].

References

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