What is the Optimal Railway Network?

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We present a straightforward algorithm to optimise a railway network. Starting from some assumptions and a basic understanding of the mathematics, we show that the network can be optimised by considering only network structures such as circuits. The algorithm is applied to an abstraction of the British railway network, which concludes with a novel proposal for the structure of the optimal railway network.

Model description

Consider an idealised model of a railway, where trains leave stations as soon as possible, and the frequency of train departures is as high as possible. We add the following crucial condition: trains scheduled to depart must wait for all arriving trains before departing. Let $x_i(k)$ represent the k^{th} departure time at station S_i. Then, given our conditions, the future timetable of the simple railway network of Fig. 1 is given by the following recurrence relation.

$$x_1(k+1) = \max(x_1(k)+2, x_2(k)+5) \quad (1)$$
$$x_2(k+1) = \max(x_1(k)+3, x_2(k)+3) \ . \quad (2)$$

By making operator transforms $\max \mapsto \oplus$ and $+ \mapsto \otimes$, this system is linearised, which takes

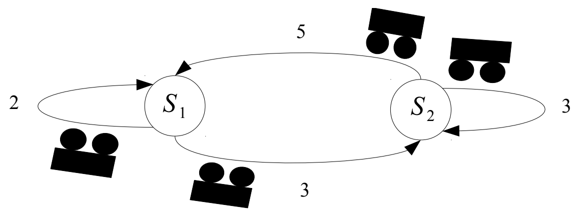

Figure 1: Simple railway network with two stations, one train on each track.

us to the realm of *max-plus algebra* (MPA). Such a system has been well-studied for the railway application in [1], where the intimate link between eigenvalues of the network adjacency matrix, railway performance, and network structure has been highlighted.

Here, we build on this idea to optimise the British railway network. We present an algorithm that takes only a standard understanding of MPA, following which the optimisation steps are grounded solely in the study of circuits in the network. A fundamental theorem of MPA says that, in a strongly connected network such as the railway, the eigenvalue is unique and determined by the circuit(s) with the largest average weight. Thus, critical circuits in the network are those circuits that comprise the longest average round-trip time λ, which also represents average inter-departure times. Reducing λ therefore addresses the aim of maximising the frequency of departures; our algorithm achieves this by identifying such circuits and incrementally adding dummy stations on them. Subsequently, new critical circuits are identified, and the process is repeated until desired.

Thus, we strategically add nodes and edges, enlarging circuits along the way so as to reduce their λ value. Ultimately, this produces a one-way ring layout as optimal (see Fig. 2). Indeed,

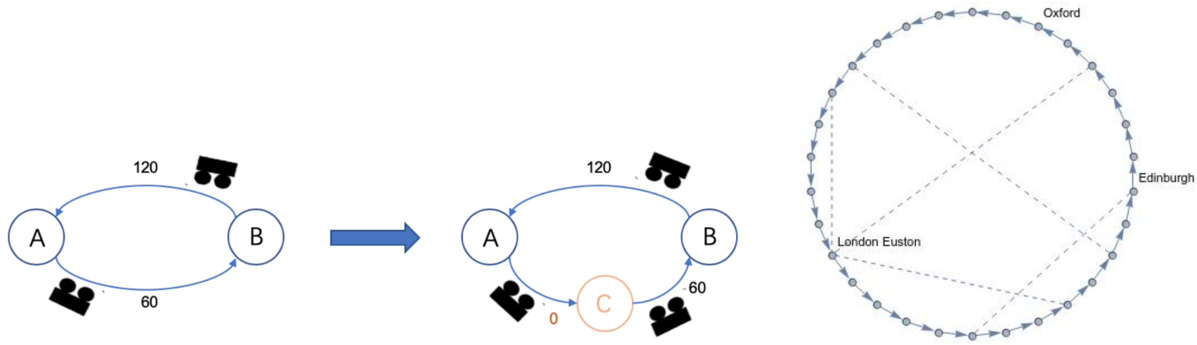

Figure 2: Left: Adding a dummy station enlarges a circuit and reduces its average weight: from $\lambda = (180)/2 = 90$ to $\lambda = 180/3 = 60$. Right: the optimal railway network is a one-way ring?

this allows stations to wait only for one train before departures, though it looks impractical; for instance, to travel from London Euston to Oxford in the south, it seems one would have to go through Edinburgh, a futile diversion to the north. However, the additional nodes (the dummy stations) in this layout represent platforms within stations, and the dashed edges represent walks between these platforms. This is what produces the impracticality illusion; we have in fact split London Euston into multiple substations, i.e., platforms.

Implications

Max-plus algebra has been proposed as a useful modelling tool for optimising railway timetables in [1] as well as for the scheduling in certain biological applications and manufacturing processes [2, 3]. However, there has been minimal focus on the structure of the network. Here we show that we can optimise the railway structure at little cost. We envisage beneficial discussions with experts at CCS 2019 to further this research, particularly in terms of wider implications and other applications; does this ring network arise in biological networks and manufacturing?

References

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